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DYNAMIC PROBLEM OF THE THEORY OF ELASTICITY FOR A PLANE CONTAINING A RIGID CRUCIFORM INCLUSION†

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The oscillations of a rigid cruciform symmetric inclusion lying in an unbounded medium and to which a time-periodic twisting moment is applied are considered. (The plane deformation case is treated.) Discontinuous solutions of plane elasticity theory are used (in which displacements and stresses are discontinuous along a given line). A system of integral equations for the required discontinuities is obtained and solved by the mechanical-quadrature method. The frequency-dependence of the inclusion oscillation amplitude is investigated together with the elastic stress density near its ends and the wave field in the far zone.

1. We will construct a discontinuous solution of a dynamical problem in the theory of elasticity for the case of harmonic oscillations of a medium under plane strain conditions. The discontinuities lie in the range $x=0$, $-a_1 \leqslant y \leqslant a_1$, with jumps (here and henceforth the factor $e^{-i\omega t}$ is omitted)

$$\langle \sigma_x \rangle = \chi_1(y),\ \langle \tau_{xy} \rangle = \chi_2(y),\ \langle u \rangle = \chi_3(y)$$
$$\langle v \rangle = \chi_4(y),\quad \langle f \rangle = f(+0,y) - f(-0,y) \tag{1.1}$$

The discontinuous solution of the Lamé equations for harmonic oscillations under plane deformation conditions with discontinuities (1.1) and satisfying the radiation condition at infinity is the function

$$\begin{aligned}
u_1(x,y) &= \int_{-a_1}^{+a_1} \frac{\chi_1(\eta)}{\mu\kappa_2^2}\Big[\Big(\kappa_1^2 + \frac{\partial^2}{\partial y^2}\Big) r_1 - \frac{\partial^2 r_2}{\partial y^2}\Big] d\eta + \int_{-a_1}^{+a_1} \frac{\chi_2(\eta)}{\mu\kappa_2^2} \frac{\partial^2}{\partial x \partial y}(r_2 - r_1) d\eta + \\
&+ \frac{\partial}{\partial x}\int_{-a_1}^{+a_1} \chi_3(\eta) r_1 d\eta + 2\frac{\partial}{\partial x}\int_{-a_1}^{+a_1}\frac{\chi_3(\eta)}{\kappa_2^2}\frac{\partial^2}{\partial y^2}(r_1 - r_2) d\eta + \\
&+ \int_{-a_1}^{+a_1}\frac{\chi_4(\eta)}{\kappa_2^2}\Big[2\Big(\kappa_1^2 + \frac{\partial^2}{\partial y^2}\Big)\frac{\partial r_1}{\partial y} - \Big(\kappa_2^2 + 2\frac{\partial^2}{\partial y^2}\Big)\frac{\partial r_2}{\partial y}\Big] d\eta \\
v_1(x,y) &= \int_{-a_1}^{+a_1}\frac{\chi_1(\eta)}{\mu\kappa_2^2}\frac{\partial^2}{\partial x\partial y}(r_2 - r_1) d\eta + \int_{-a_1}^{+a_1}\frac{\chi_2(\eta)}{\mu\kappa_2^2}\Big[-\frac{\partial^2 r_1}{\partial y^2} + \\
&+ \Big(\kappa_2^2 + \frac{\partial^2}{\partial y^2}\Big) r_2\Big] d\eta + \int_{-a_1}^{+a_1}\frac{\chi_3(\eta)}{\kappa_2^2}\Big[\Big(\kappa_2^2 + 2\frac{\partial^2}{\partial y^2}\Big)\frac{\partial r_1}{\partial y} - 2\Big(\kappa_2^2 + \\
&+ \frac{\partial^2}{\partial y^2}\Big)\frac{\partial r_2}{\partial y}\Big] d\eta + \frac{\partial}{\partial x}\int_{-a_1}^{+a_1}\chi_4(\eta) r_2 d\eta + 2\frac{\partial}{\partial x}\int_{-a_1}^{+a_1}\frac{\chi_4(\eta)}{\kappa_2^2}\frac{\partial^2}{\partial y^2}(r_2 - r_1) d\eta
\end{aligned} \tag{1.2}$$

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Here

$$\kappa_1^2 = \rho\omega^2/(\lambda + 2\mu), \quad \kappa_2^2 = \rho\omega^2/\mu$$

$$r_j = r_j(\eta - y, x) = -\frac{1}{4\pi^2}\int\int_{-\infty}^{+\infty} \frac{\exp(-i\alpha x + i\beta(\eta - y))}{\alpha^2 + \beta^2 - \kappa_j^2} d\alpha d\beta =$$

$$= -\frac{1}{4} iH_0^{(1)}[\kappa_j\sqrt{(\eta - y)^2 + x^2}], \quad j = 1, 2$$

and $r_j(y, x)$ is a solution of the Helmholtz equation: $\Delta\varphi_j + \kappa_j^2\varphi_i = \delta(x)\delta(y)$.

The generalized version of the method of integral transformations [1] was applied to construct the discontinuous solution.

We consider the discontinuous solution that undergoes the jumps

$$[\sigma_y] = \varphi_1(x), \quad [\tau_{yx}] = \varphi_2(x), \quad [v] = \varphi_3(x), \quad [u] = \varphi_4(x)$$
$$[f] = f(x, +0) - f(x, -0), \quad -a_2 \leqslant x \leqslant a_2$$

in the interval $y = 0$, $-a_2 \leqslant x \leqslant a_2$.

We denote it by $u_2(x, y)$ and $v_2(x, y)$. It can then be constructed from formulae (1.2) if $\chi_j(x)$ is replaced by $\varphi_j(x)$ and the variables x and y are interchanged. Here the formula for u_1 becomes a formula for v_2 and the formula for v_1 becomes a formula for u_2.

The discontinuous solutions constructed can be effectively used to reduce problems in the theory of elasticity for media containing crack-type defects and thin rigid inclusions to integral equations.

2. We consider the following problem. Suppose that a thin rigid cruciform inclusion is situated in an elastic medium and occupies two segments intersecting at the origin of coordinates

$$x = 0, \quad -a_1 \leqslant y \leqslant a_1, \quad y = 0, \quad -a_2 \leqslant x \leqslant a_2$$

to which a moment $Me^{-i\omega t}$ varying periodically with time is applied. The inclusion will be modelled by rectilinear segments on which the stresses are discontinuous

$$\langle \sigma_x \rangle = \chi_1(y), \quad \langle \tau_{xy} \rangle = \chi_2(y), \quad -a_1 \leqslant y \leqslant a_1$$
$$[\sigma_y] = \varphi_1(x), \quad [\tau_{yx}] = \varphi_2(x), \quad -a_2 \leqslant x \leqslant a_2 \tag{2.1}$$

while the displacements satisfy the conditions

$$u(\pm 0, y) = \gamma y, \quad v(\pm 0, y) = 0, \quad -a_1 \leqslant y \leqslant a_1$$
$$v(x, \pm 0) = \gamma x, \quad u(x, \pm 0) = 0, \quad -a_2 \leqslant x \leqslant a_2 \tag{2.2}$$

where γ is the angle of rotation of the inclusion under the action of the applied moment.

From symmetry one can show that there are no shear stresses in the contact domain of the inclusion and medium, i.e. $\chi_2(y) = 0$, $\varphi_2(\chi) = 0$, and the discontinuities in $\chi_1(y)$ and $\varphi_1(x)$ are odd. Here the condition that the corresponding displacements in (2.2) must be zero is satisfied automatically.

We will look for the solution of the problem in the form of the sum of two discontinuous solutions

$$u = u_1 + u_2, \quad v = v_1 + v_2 \tag{2.3}$$

constructed from formulae (1.2), where one must put $\chi_j(y) = 0$, $\varphi_j(x) = 0$, $j = 2, 3, 4$. It has the form

$$
\begin{aligned}
u(x,y) &= \int_{-a_1}^{+a_1} \frac{\chi_1(\eta)}{\mu\kappa_2^2} \left[\left(\kappa_1^2 + \frac{\partial^2}{\partial y^2}\right) r_1(\eta - y, x) - \frac{\partial^2}{\partial y^2} r_2(\eta - y, x)\right] d\eta + \\
&+ \int_{-a_2}^{+a_2} \frac{\varphi_1(\eta)}{\mu\kappa_2^2} \left[\frac{\partial^2}{\partial x \partial y} r_2(\eta - x, y) - \frac{\partial^2}{\partial x \partial y} r_1(\eta - x, y\right] d\eta \\
v(x,y) &= \int_{-a_1}^{a_1} \frac{\chi_1(\eta)}{\mu\kappa_2^2} \left[\frac{\partial^2}{\partial x \partial y} r_2(\eta - y, x) - \frac{\partial^2}{\partial x \partial y} r_1(\eta - y, x)\right] d\eta + \\
&+ \int_{-a_2}^{+a_2} \frac{\varphi_1(\eta)}{\mu\kappa_2^2} \left[\left(\kappa_1^2 + \frac{\partial^2}{\partial x^2}\right) r_1(\eta - x, y) - \frac{\partial^2}{\partial x^2} r_2(\eta - x, y)\right] d\eta
\end{aligned}
\tag{2.4}
$$

The function $u(x, y)$ is odd in the variable y, which $v(x, y)$ is odd in the variable x.

To determine the required discontinuities $\chi_1(\eta)$ and $\varphi_1(\eta)$ from the remaining conditions in (2.1) one can obtain integral equations. It is more convenient not to use conditions (2.2) themselves, but equivalent conditions obtained by differentiating the former

$$
u'_y(\pm 0, y) = \gamma, \quad -a_1 \leqslant y \leqslant a_1, \quad v'_x(x, \pm 0) = \gamma, \quad -a_2 \leqslant x \leqslant a_2 \tag{2.5}
$$

Substituting (2.4) into (2.5), we arrive at a system of two integral equations, which after reduction to the interval [–1, 1] have the form

$$
\begin{aligned}
&\frac{1}{2\pi} \int_{-1}^{+1} g_1(\tau) \left[\frac{1 + \xi^2}{\tau - t} + R(\tau - t) + iS(\tau - t)\right] d\tau + \\
&+ \frac{\epsilon}{2\pi} \int_{-1}^{+1} g_2(\tau) \left[-(1 - \xi^2) P(\epsilon\tau, t) + Q(\epsilon\tau, t) + iG(\epsilon\tau, t)\right] d\tau = -1 \\
&\frac{1}{2\pi} \int_{-1}^{+1} g_1(\tau) \left[-(1 - \xi^2) P(\tau, \epsilon t) + Q(\tau, \epsilon t) + iG(\tau, \epsilon t)\right] d\tau + \\
&+ \frac{1}{2\pi} \int_{-1}^{+1} g_2(\tau) \left[\frac{1 + \xi}{\tau - t} + \epsilon R(\epsilon(\tau - t)) + i\epsilon S(\epsilon(\tau - t))\right] d\tau = -1
\end{aligned}
\tag{2.6}
$$

Here

$$
g_1(\tau) = \frac{\chi_1(a_1\tau)}{\mu\gamma}, \quad g_2(\tau) = \frac{\varphi_1(a_2\tau)}{\mu\gamma}, \quad P(\tau, t) = \frac{\tau(\tau^2 - t^2)}{(\tau^2 + t^2)^2}
$$

$$
\epsilon = \frac{a_2}{a_1}, \quad \xi^2 = \frac{1 - 2\nu}{2(1 - \nu)}, \quad R(z) = R_1(z) + R_2(z)
$$

$$
R_1(z) = 2\xi^3\left(c_0 + \ln\frac{\xi\kappa_0 |z|}{2}\right) \Sigma_\alpha(\xi z) - \frac{\xi^3}{2} \Sigma_\beta(\xi z)
$$

$$
R_2(z) = \left(c_0 + \ln\frac{\kappa_0 |z|}{2}\right)\left[-2\Sigma_\alpha(z) + 4\Sigma_\beta(z)\right] - 2\Sigma_\lambda(z) + \Sigma_\delta(z)
$$

$$
S(z) = \pi\left[-2\Sigma_\alpha(z) - \xi^3\Sigma_\alpha(\xi z) + \Sigma_\beta(z)\right]
$$

$$
Q(x, y) = Q_1(x, y) + Q_2(x, y)
$$

$$Q_1(x,y) = 2\xi^3 (c_0 + \ln\frac{\xi\kappa_0 p}{2}) [A_1(x,y)\,\Sigma_\alpha(\xi p) + A_2(x,y)\,\Sigma_\beta(\xi p)] +$$
$$+ \xi^3 [-A_2(x,y)\,\Sigma_\lambda(\xi p) + A_1(x,y)\,\Sigma_\delta(\xi p)]$$
$$Q_2(x,y) = -2(c_0 + \ln\frac{\kappa_0 p}{2}) [A_1(x,y)\,\Sigma_\alpha(p) + A_2(x,y)\,\Sigma_\beta(p)] +$$
$$+ A_2(x,y)\,\Sigma_\lambda(p) - A_1(x,y)\,\Sigma_\delta(p)$$
$$G(x,y) = \pi\,\{ -\xi^3 [A_1(x,y)\,\Sigma_\alpha(\xi p) + A_2(x,y)\,\Sigma_\beta(\xi p)] -$$
$$- A_1(x,y)\,\Sigma_\alpha(p) + A_2(x,y)\,\Sigma_\beta(p)\,\}$$
$$\Sigma_\alpha(z) = \sum_{k=1}^{\infty} \alpha_k z^{2k-1}, \quad p = \sqrt{x^2+y^2}$$
$$A_1(x,y) = \frac{x(x^2-3y^2)}{p^3}, \quad A_2(x,y) = \frac{4xy^2}{p^3}$$
$$\alpha_k = \frac{\beta_k}{k+1}, \quad \beta_k = \frac{(1)^k \kappa_0^{2k}}{k!\,(k-1)!\,2^{2k}}, \quad \lambda_k = \beta_k(2h_{k-1} + \frac{1}{k})$$
$$\delta_k = \frac{\beta_k}{k}\,[4h_k - \frac{1}{k+1}(2h_k + \frac{1}{k+1})], \quad h_0 = 1, \quad h_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$$
$$\kappa_0 = \kappa_2 a_1, \quad c_0 = 0.5772157$$

(where v is Poisson's ratio).

To determine the required constant it is necessary to use the equation of motion of the inclusion as a rigid body

$$e^{-i\omega t} M = e^{-i\omega t} M_r + J_z \epsilon_0, \quad M_r = \int_{-a_1}^{+a_1} y\chi_1(y)\,dy + \int_{-a_2}^{+a_2} x\varphi_1(x)\,dx$$

where J_z is the moment of inertia of the inclusion, ϵ_0 is the angular acceleration and M_t is the moment of elastic reaction forces.

We transform the equation of motion into the form

$$M_* = \gamma\,[\int_{-1}^{+1} t g_1(t)\,dt + \epsilon^2 \int_{-1}^{+1} t g_2(t)\,dt] - \gamma\kappa_0^2\beta$$
$$M_* = \frac{M}{\mu a_1^2}, \quad \beta = \frac{2m_1}{3\rho a_1^2}(1+\epsilon^2) \tag{2.7}$$

where m_1 is the mass of the part of the inclusion occupying the interval [–1, 1] and p is the density of the elastic medium.

We will construct the solution of the system of integral equations (2.6) numerically using the method of mechanical quadratures [2, 3] and the oddness of the functions $g_1(t)$ and $g_2(t)$. To this end we represent the required functions in the form

$$g_i(t) = (1 - t^2)^{-1/2}\psi_i(t), \quad i = 1, 2 \tag{2.8}$$

and approximate the $\psi_i(t)$ by odd interpolating polynomials of degree $2n-1$ constructed with respect to the nodes

$$t_l = \cos x_l, \quad x_l = (2l-1)\pi/(4n), \quad l = 1, 2, \ldots, 2n$$

These polynomials have the form [4]

$$\psi_i(t) = L_{2n-t}^{(i)}(t) = \frac{2}{n}\sum_{l=1}^{n}\psi_i(t_l)\sum_{m=1}^{n}\cos(2m-1)x_l T_{2m-1}(t) \tag{2.9}$$

where the $T_{2m-1}(t)$ are Chebyshev polynomials.

Then the following quadrature formulae can be obtained for the singular integral operators in (2.6)

$$\int_{-1}^{+1}\frac{g_i(\tau)}{\tau - t_j}\,d\tau = 2\pi\sum_{l=1}^{n}A_{jl}\,\psi_j(t_l),\quad j = 1,2,\ldots,n$$
$$A_{jl} = \frac{1}{n}\sum_{m=1}^{n}\frac{\cos(2m-1)x_l\sin(2m-1)x_j}{\sin x_l} \tag{2.10}$$

$$\int_{-1}^{+1} g_i(\tau)P_i(\tau, t_j)\,d\tau = 2\pi\sum_{l=1}^{n}B_{jl}^{(i)}\psi_i(t_l) \tag{2.11}$$
$$P_1(\tau, t) = P(\epsilon\tau, t),\quad P_2(\tau, t) = P(\tau, \epsilon t)$$
$$B_{jl}^{(i)} = \frac{p_{ij}^{-3}}{n}\sum_{m=1}^{n}(-1)^m q_{ij}^{2m-1}b_m^{(i)}\cos(2m-1)x_l$$
$$p_{1j} = \sqrt{\epsilon^2 + t_j^2},\quad p_{2j} = \sqrt{1+\epsilon^2 t_j^2}$$
$$q_{1j} = \epsilon(p_{1j} + t_j)^{-1},\quad q_{2j} = (p_{2j} ++ t_j)^{-1}$$
$$b_m^{(1)} = (2m-1)t_j p_{1j} - \epsilon^2,\quad b_m^{(2)} = (2m-1)\epsilon t_j p_{2j} - 1$$

Replacing the singular integrals in (2.6) by the quadrature formulae (2.10) and (2.11), and the regular integrals by Gaussian quadrature formulae [4], and equating the left- and right-hand sides for $t = t_j$ ($j = 1, 2, \ldots, n$), we obtain a system of linear algebraic equations in the $\psi_i(t_l)$ ($i = 1, 2$, $l = 1, 2, \ldots, n$)

$$\sum_{l=1}^{n}\left[2(1+\xi^2)A_{jl} + \frac{R_{jl}^{(1)} + iS_{jl}^{(1)}}{2n}\right]\psi_1(t_k) + \sum_{l=1}^{n}\left[-2(1-\xi^2)B_{jl}^{(2)} + \right.$$
$$\left. + \frac{Q(\epsilon t_l, t_j) + iG(\epsilon t_l, t_j)}{n}\right]\psi_2(t_l) = -4 \tag{2.12}$$
$$\sum_{l=1}^{n}\left[-2(1-\xi^2)B_{jl}^{(1)} + \frac{Q_{lj}^{(1)} + iG_{lj}^{(1)}}{n}\right]\psi_1(t_l) + \sum_{l=1}^{n}\left[2(1+\xi^2)A_{jl} + \right.$$
$$\left. + \epsilon\frac{R_{jl}^{(2)} + iS_{jl}^{(2)}}{2n}\right]\psi_2(t_l) = -4$$
$$R_{lj}^{(i)} = R[\gamma_i(t_l - t_j)] - R[\gamma_i(t_l + t_j)],\quad S_{lj}^{(i)} = S[\gamma_l(t_l - t_j)] - S[\gamma_i(t_l + t_j)]$$
$$i = 1, 2,\quad \gamma_1 = 1,\quad \gamma_2 = \epsilon$$
$$Q_{lj}^{(1)} = Q(\epsilon t_l, t_j),\quad Q_{lj}^{(2)} = Q(t_l, \epsilon t_j),\quad G_{lj}^{(1)} = G(\epsilon t_l, t_j),\quad G_{lj}^{(2)} = G(t_l, \epsilon t_j)$$

Condition (2.7) for determining γ acquires the form

$$M_* = \gamma\frac{\pi}{n}\sum_{l=1}^{n}t_l[\psi_1(t_l) + \epsilon^2\psi_2(t_l)] - \gamma\kappa_0^2\beta \tag{2.13}$$

Solving system (2.12), (2.13) using formulae (2.9) and (2.8), we construct an approximate solution of the system of integral equations (2.6).

3. To describe the elastic stress density near the inclusion we introduce the stress intensity factor (SIF) [5]

$$K^{II}(a_1) = \lim_{y\to a_1+0} \sqrt{2(y-a_1)}\,\tau_{xy}(0, y), \quad K^{II}(a_2) = \lim_{y\to a_2+0} \sqrt{2(x-a_2)}\,\tau_{yx}(x, 0)$$

The SIF is expressed in terms of the approximate solution of the system of integral equations obtained by the following formulae

$$K^{II}(a_j) = \mu\sqrt{a_j}\,k_j, \quad k_j = -\frac{\xi^2\gamma}{n}\sum_{l=1}^{n} \psi_j(t_l)\frac{(-1)^l}{\sin x_l}, \quad j = 1, 2 \tag{3.1}$$

To describe the wave field far from the inclusion we will obtain an asymptotic formula for the displacements $u(x, y)$ and $v(x, y)$. We change to polar coordinates $x = R\cos\theta$ and $y = R\sin\theta$ in (2.4) and let $R\to\infty$. Using the asymptotic expansion of the Hankel function together with the approximate solution of the system of integral equations (2.6) we find

$$u_*(R, \theta) = E_1 f_1(\theta) + E_2 f_2(\theta) + O(R_0^{-3/2})$$
$$v_*(R, \theta) = -E_1 p_1(\theta) + E_2 p_2(\theta) + O(R_0^{-3/2}) \tag{3.2}$$
$$E_1 = \xi^{3/2}\eta\exp\left[i\xi\left(R_0 - \frac{\pi}{4}\right)\right], \quad E_2 = \eta\exp\left[i\left(R_0 - \frac{\pi}{4}\right)\right], \quad \eta = \frac{\gamma}{2}\sqrt{\frac{2}{\pi R_0}}$$
$$f_k(\theta) = \sigma_{k1}\cos^2\theta - \epsilon\sigma_{k2}\sin\theta\cos\theta \quad p_k(\theta) = \sigma_{k1}\sin\theta\cos\theta - \epsilon\sigma_{k2}\sin^2\theta$$
$$\sigma_{kj} = \frac{1}{2\pi}\int_{-1}^{+1} g_j(\tau)\exp(-ib_k\kappa_0\cos\theta)\,d\tau, \quad k = 1, 2, \quad j = 1, 2$$
$$b_1 = \epsilon, \quad b_2 = 1, \quad u_*(R, \theta) = a_1^{-1}u(R\cos\theta, R\sin\theta), \quad v_*(R, \theta) = a_1^{-1}v(R\cos\theta, R\sin\theta)$$

Using the approximate solution constructed, formulae (2.13) and (3.1) were used to investigate the dependence of the maximum amplitude of the inclusion oscillations $|\gamma|$ and the maximum absolute values of the SIF $|k_1|$, $|k_2|$ on the parameter h_0 for $\nu = 0.25$, $M_* = 1$, $\beta = 2$. These dependencies are

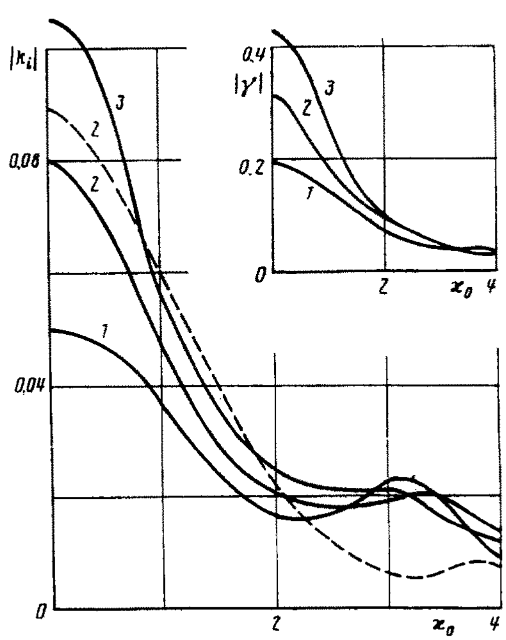

FIG. 1.

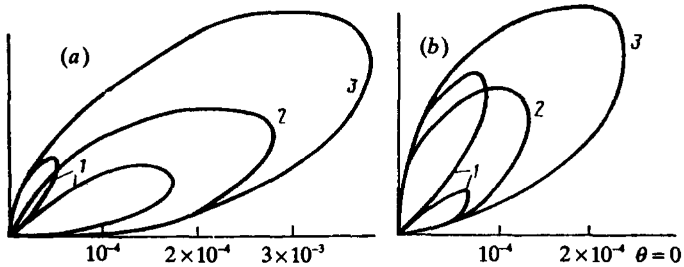

FIG. 2.

shown in Fig. 1. Curve 1 corresponds to an equal-sided cruciform inclusion (here $k_1=k_2$), curve 2 corresponds to a ratio between the sides of $\epsilon=0.5$, and curve 3 to the case of a single rectilinear inclusion ($\epsilon=0$). The solid curve shows the variation of $|k_1|$ and the dashed one shows $|k_2|$. It is clear that as κ_0 increases, the quantity $|\gamma|$ decreases to some value, and then it stabilizes, with all three curves almost coinciding.

As κ_0 increases the SIF up to a certain instant decreases monotonically, and then begins to oscillate. For low oscillation frequencies (i.e. for small κ_0), the largest stress density is near the rectilinear inclusion, and then as κ_0 increases all the curves become close and intersect one another.

The wave field far from the centre of the inclusion was also investigated. Figures 2(a) and (b) show the dependence of the maximum absolute values of the displacements $|u_*|$ and $|v_*|$ on the polar angle $0\leq\theta\leq\pi/2$ for $R_0=1000$, $\kappa_0=3$. The notation is the same as in Fig. 1.

REFERENCES

1. POPOV G. Ya., A method of solving mechanical problems for a domain with slits or thin inclusions. *Prikl. Mat. Mekh.* **42**, 122–135, 1978.
2. KALANDIYA A. I., *Mathematical Methods of Two-dimensional Elasticity.* Nauka, Moscow, 1973.
3. BELOTSERKOVSKII S. M. and LIFANOV I. K., *Numerical Methods for Singular Integral Equations.* Nauka, Moscow, 1985.
4. KRYLOV V. I., *Approximate Evaluation of Integrals.* Nauka, Moscow, 1967.
5. YEVTUSHENKO A. A. and PAUK V. I., Influence of material inhomogeneities on the stress distribution near a thin elastic inclusion. *Prikl. Mat. Mekh.* **53**, 651–657, 1989.

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